SOLUTIONS OF EXERCISES.

162

A BODY is projected at an angle of 30° with the horizon, with a given velocity. Determine the constant resistance it must suffer in the direction contrary to its motion in order that it may come to rest when it returns to the horizontal plane whence it started. Also determine the horizontal range, time of flight, and length of trajectory.

[*Jas. M. Ingalls.*]

SOLUTION.

Let V be the initial velocity, α the angle of projection, φ the inclination at a point of the trajectory where the velocity is v, U and u the corresponding *horizontal* velocities, r the constant resistance, and w the weight of the projectile. Put $r/w = n$. Then we have

$$\frac{du}{u} = n\frac{d\varphi}{\cos\varphi};$$

whence

$$\log u = n \log \tan\left[\frac{1}{4}\pi + \frac{1}{2}\varphi\right] + C.$$

Let v_0 be the velocity when $\varphi = 0$, that is, at the summit of the trajectory; then $C = \log v_0$, and we have

$$u = v_0 \tan^n\left[\frac{1}{4}\pi + \frac{1}{2}\varphi\right].$$

$$\therefore v_0 = u \cot^n\left[\frac{1}{4}\pi + \frac{1}{2}\varphi\right] = U \cot^n\left[\frac{1}{4}\pi + \frac{1}{2}\alpha\right]. \qquad (1)$$

Substituting the above value of u in the differential equations

$$dt = -\frac{u}{g}\sec^2\varphi d\varphi,$$

$$dx = -\frac{u^2}{g}\sec^2\varphi d\varphi,$$

$$dy = -\frac{u^2}{g}\tan\varphi \sec^2\varphi d\varphi,$$

$$ds = -\frac{u^2}{g}\sec^3\varphi d\varphi,$$

integrating between the limits a and φ, substituting for v_0 its value from (1), and reducing, we have (supposing $n > 1$) the following general equations:

$$t = \frac{V}{g}\frac{n - \sin a}{n^2 - 1} - \frac{v}{g}\frac{n - \sin\varphi}{n^2 - 1},$$

$$x = \frac{V^2}{g}\frac{(2n - \sin a)\cos a}{4n^2 - 1} - \frac{v^2}{g}\frac{(2n - \sin\varphi)\cos\varphi}{4n^2 - 1},$$

$$y = \frac{V^2}{g}\frac{(2n - \sin a)\sin a - 1}{4(n^2 - 1)} - \frac{v^2}{g}\frac{(2n - \sin\varphi)\sin\varphi - 1}{4(n^2 - 1)},$$

$$s = \frac{V^2}{g}\frac{2n(n - \sin a) - \cos^2 a}{4n(n^2 - 1)} - \frac{v^2}{g}\frac{2n(n - \sin\varphi) - \cos^2\varphi}{4n(n^2 - 1)}.$$

From (1) we have

$$v = v_0 \sec\varphi \tan^n\left[\frac{1}{4}\pi + \frac{1}{2}\varphi\right],$$

which when $n > 1$, reduces to zero when $\varphi = -\frac{1}{2}\pi$. Therefore when the constant resistance is greater than the weight of the projectile, the velocity continually decreases, and becomes zero when the last element of the trajectory is vertical. At this point we have

$$t_{-\frac{1}{2}\pi} = \frac{V}{g}\frac{n - \sin a}{n^2 - 1},$$

$$x_{-\frac{1}{2}\pi} = \frac{V^2}{g}\frac{(2n - \sin a)\cos a}{4n^2 - 1},$$

$$y_{-\frac{1}{2}\pi} = \frac{V^2}{g}\frac{(2n - \sin a)\sin a - 1}{4(n^2 - 1)},$$

$$s_{-\frac{1}{2}\pi} = \frac{V^2}{g}\frac{2n(n - \sin a) - \cos^2 a}{4n(n^2 - 1)}.$$

To determine the relation that must exist between n and a, in order that the velocity of the projectile may be zero when it returns to the horizontal plane passing through the point of departure, we must make $y_{-\frac{1}{2}\pi} = 0$; which gives

$$\sin a\,(2n - \sin a) = 1,$$

or

$$n = \frac{1 + \sin^2 a}{2\sin a}.$$

In exercise 162, $a = 30°$; whence $n = \frac{5}{4}$. That is, the resistance is one and one-fourth the weight of the projectile. Substituting these values of a and n in the above equations, we have

$$T = \frac{4}{3}\frac{V}{g}; \qquad X = \frac{4\sqrt{3}}{21}\frac{V^2}{g}; \qquad S = \frac{2}{5}\frac{V^2}{g}.$$

For the summit we have from the general expression for y, making $\varphi = 0$,

$$y_0 = \frac{4\,v_0^{\,2}}{9g}.$$

But from (1) we have

$$v_0^{\,2} = V^2 \cos^2 30° \cot^{\frac{5}{2}} 60° = \frac{V^2}{4 \times 3^{\frac{1}{4}}};$$

$$\therefore y_0 = \frac{1}{9 \times 3^{\frac{1}{4}}}\,\frac{V^2}{g}.$$

[*Jas. M. Ingalls.*]

282

The centre and three points do not determine a conic when two of the three points are at the extremities of a diameter ($2k$). [*R. H. Graves.*]

SOLUTION.

If the centre be taken as the origin of co-ordinates, and $(+\,c, 0)$ and $(-\,c, 0)$ be two given points, the equation to the conic is

$$x^2 + Bxy + Cy^2 = c^2.$$

One more point cannot determine B and C. [*W. M. Thornton.*]

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If $\varkappa$, λ, μ, ν be any four angles, and if

$$\varkappa + \lambda + \mu + \nu = 2\sigma,$$

prove

$$\begin{aligned}&\sin\varkappa \sin\lambda \sin\mu \sin\nu + \cos\varkappa \cos\lambda \cos\mu \cos\nu \\ &\quad = \sin(\sigma - \varkappa)\sin(\sigma - \lambda)\sin(\sigma - \mu)\sin(\sigma - \nu) \\ &\quad + \cos(\sigma - \varkappa)\cos(\sigma - \lambda)\cos(\sigma - \mu)\cos(\sigma - \nu).\end{aligned}$$

[*Yale.*]

SOLUTION.

Expressing $\sin\varkappa \sin\lambda$, etc. as functions of sums and differences of angles, we obtain

$$\begin{aligned}&\sin\varkappa \sin\lambda \sin\mu \sin\nu + \cos\varkappa \cos\lambda \cos\mu \cos\nu \\ &\qquad = \tfrac{1}{2}\cos(\varkappa + \lambda)\cos(\mu + \nu) + \tfrac{1}{2}\cos(\varkappa - \lambda)\cos(\mu - \nu).\end{aligned}$$

If we put $x + \lambda = 2\sigma - \mu - \nu$ and $\mu + \nu = 2\sigma - x - \lambda$, it is readily seen that this expression will not change its value by substituting $\sigma - x$ for x, etc.

[*Ormond Stone.*]

313

GIVEN three points and three straight lines in a plane, the determinant of the nine perpendiculars from the points to the lines is equal to twice the product of the areas of the triangles formed by the points and by the lines, divided by the radius of the circle circumscribing the latter.

[*W. W. Johnson.*]

SOLUTION.

Let the triangle formed by the three lines be the triangle of reference and the co-ordinates of the three points be $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; $\alpha_3, \beta_3, \gamma_3$. To prove

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \frac{2\Delta\Delta'}{R}.$$

Δ', the area of the triangle formed by the three points, is

$$\frac{1}{2S}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix},$$

where $S = \alpha \sin A + \beta \sin B + \gamma \sin C$ (see Whitworth's Modern Analytic Geometry, p. 22), $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, and $a\alpha + b\beta + c\gamma = 2\Delta$; whence the theorem.

[*H. B. Newson.*]

318

D is any point in the base BC of a triangle ABC. O_1 and O_2 are the centres of the inscribed circles of BAD and CAD. Then, if $\triangle$ = area ABC,

(1) Area $AO_1O_2 = \frac{r\triangle}{a} \operatorname{cosec} ADC$;

(2) If AD bisects A, $AO_1O_2 = \frac{r\triangle}{b+c} \operatorname{cosec} \frac{1}{2}A$;

(3) If AD is the altitude, $AO_1O_2 = \frac{r\triangle}{a}$.

[*T. U. Taylor.*]

SOLUTION.

(1) $AO_1 = \frac{c \sin \frac{1}{2}B}{\sin(\frac{1}{2}B + BAO_1)}$, $AO_2 = \frac{b \sin \frac{1}{2}C}{\sin(\frac{1}{2}C + CAO_2)}$;

$$\begin{aligned}\therefore \text{ area } AO_1O_2 &= \tfrac{1}{2}AO_1 \,.\, AO_2 \sin O_1AO_2\\ &= \tfrac{1}{2}\frac{bc \sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C}{\sin(\frac{1}{2}B + BAO_1)\sin(\frac{1}{2}C + CAO_2)}\\ &= \frac{r\triangle}{2a \sin\frac{1}{2}ADC \cos\frac{1}{2}ADC}\\ &= \frac{r\triangle}{a}\operatorname{cosec} ADC.\end{aligned}$$

(2) If AD bisects A, then $ADC = 90° - \frac{1}{2}(C - B)$. By substituting this in (1) it easily reduces to the required form.

(3) If AD is the altitude, $ADC = 90°$;

$$\therefore\ AO_1O_2 = \frac{r\triangle}{a}.$$

[*T. U. Taylor.*]

319

To construct a triangle ABC ($BC > AB > CA$), the angle A being known, and having given

(1) $AB + BC = m$, $AB + CA = n$;

or (2) $BC - AB = p$, $AB - CA = q$. [*J. E. Hendricks.*]

SOLUTION.

1. Draw an indefinite right line MN, and through any point D draw PP', making an angle with MN equal to the given angle at A. From D toward M set off $DB = m$, and from B toward N set off $BF = n$. With D as centre and radius DF describe a semicircle EFE' and join EB and EF. With F as centre and radius DF describe an arc cutting EB in G, and join FG. Through B parallel with FG draw BC meeting EF produced in C, and through C draw CA parallel with PP' and intersecting BD in A; then is ABC the required triangle.

Because the triangle FDE is isosceles the similar triangle FAC is isosceles, therefore $AC = AF$. Through C draw CH parallel with AD and intersecting PP' in H. Then, by similar triangles, we have,

$$EF : FD :: EC : CH.$$

Also $$EF : FG :: EC : CB.$$

But $FG = FD$ by construction; $\therefore\ CB = CH = AD$.

2. Let MN and PP' be drawn as in 1. From D towards M take $DB' = p$, and from B' towards M take $B'F = q$. (Because $m - n = p + q$, F occupies the same position on the line MN in both cases.) With D as centre and radius DF describe the semicircle EFE', as in the construction of 1, and join $E'B'$ and $E'F$. With F as centre and radius DF describe an arc cutting $B'E'$ in G', and join FG'. Through B' parallel with FG' draw $B'C'$ meeting $E'F$ produced in C' and draw $C'A$ parallel with PP' and intersecting DB in A; then is $AB'C'$ the required triangle.

Because the triangle FDE' is isosceles the similar triangle FAC' is isosceles, therefore $AC' = AF$. Through C' draw $C'H'$ parallel with AD and intersecting PP' in H'. Then, by similar triangles, we have,

$$E'F : FD : : E'C' : C'H'.$$

Also $$E'F : FG' : : E'C' : C'B'.$$

But $FG' = FD$ by construction; $\therefore$ $C'B' = C'H' = AD$.

[*J. E. Hendricks.*]

EXERCISES.

329

WHAT relations must subsist between the lengths of the edges of a tetrahedron in order that the perpendiculars from the vertices to the opposite sides may meet in a common point? [*Yale Prize Problem.*]

330

FIND the sum of the series

$$1^2 + 3^2 + 6^2 + 10^2 + 15^2 + \ldots + [\tfrac{1}{2}n(n+1)]^2.$$

[*Artemas Martin.*]

331

THE extremities of a diameter of a variable ellipse having fixed foci lie on a fixed hyperbola having the same foci; show that the extremities of the conjugate diameter lie on another hyperbola having the same foci.

[*W. Woolsey Johnson.*]

332

FOUR equianharmonic points give four triangles which have four circumcircles. Show that the inverses of any point with regard to these four circles are equianharmonic. [*Frank Morley.*]

333

SHOW that

$$\sin\theta > \theta - \frac{\theta^3}{3!} + \frac{1}{45}\left[\frac{\theta^5}{2^2} - \frac{\theta^7}{2^9} + \ldots (-)^{m+1}\frac{\theta^{2m+3}}{2^{\frac{1}{2}(m^2+9m+6)}} \pm \ldots\right];$$

the general term being the mth within the brackets. [*W. H. Echols.*]

334

FIND the necessary relation between the ten distances of five points in space. [*Yale Prize Problem.*]